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A PRACTICAL METHOD OF DETERMINING ELEMENTARY DIVISORS.

By H. T. Burgess.

So far as the writer of this brief paper is aware, no method is known for determining the elementary divisors of a given matrix $\lambda A + B$ which does not involve an extremely large amount of labor.*

It is the object of this paper to give a method which is both simple and practical. The few well-known theorems on which it is based will be stated without proof.

I. Characteristic Matrices.

Fundamental Theorem.—Let $\lambda I + B$ be any characteristic matrix whose characteristic equation is

$$\varphi(\lambda) = (\lambda - \alpha_1)^{t_1}(\lambda - \alpha_2)^{t_2} \cdots (\lambda - \alpha_\epsilon)^{t_\epsilon} \cdots (\lambda - \alpha_\eta)^{t_\eta} = 0.$$

There exists a similar characteristic matrix $\lambda I + N$ with the same elementary divisors. It is of the normal form

$$\lambda I + N = \begin{Vmatrix} \lambda I + N_{t1} & & & & \\ & \lambda I + N_t & & & \\ & & \cdots & & \\ & & & \lambda I + N_{t_\epsilon} & \\ & & & & \cdots \\ & & & & & \lambda I + N_{t_\eta} \end{Vmatrix},$$

in which all elements are zero except those in the non-overlapping principal minors indicated. Any one of these principal minors as $\lambda I + N_{t_\epsilon}$ is a characteristic matrix of order t_ϵ and has the form

$$\lambda I + N_{t_\epsilon} = \begin{Vmatrix} \lambda I + E_{e_1} & & & & \\ & \lambda I + E_{e_2} & & & \\ & & \cdots & & \\ & & & \lambda I + E_{e_\pi} & \\ & & & & \cdots \\ & & & & & \lambda I + E_{e_\tau} \end{Vmatrix},$$

in which all elements are zero except those in the non-overlapping principal minors indicated. Any one of these principal minors as $\lambda I + E_{e_\pi}$ is of order e_π and has all of its elements zero except in its principal and overprincipal diagonals, all of the elements in the principal diagonal being equal to $\lambda - \alpha_\epsilon$ and all of the elements in the over-principal diagonal, to one. Further we have the relation

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* Bôcher, Introduction to Higher Algebra, p. 272.

$$e_1 \geqq e_2 \geqq \cdots \geqq e_\pi \geqq \cdots \geqq e_\tau$$

and these e's are the exponents of the elementary divisors of $\lambda I + B$ or $\lambda I + N$ which are connected with the linear factor $\lambda - \alpha_\epsilon$.*

Theorem 1. If n is a positive integer, the matrices $(\lambda I + B)^n$ and $(\lambda I + N)^n$, where $\lambda I + B$ and $\lambda I + N$ are similar, have the same ranks.

If $\lambda I + B$ and $\lambda I + N$ are similar, there exists a non-singular matrix H such that

$$H^{-1}(\lambda I + B)H = \lambda I + N.$$

Hence

$$H^{-1}(\lambda I + B)^n H = (\lambda I + N)^n,$$

for

$$H^{-1}(\lambda I + B)^n H = H^{-1}(\lambda I + B)H \cdot H^{-1}(\lambda I + B)H \cdot \cdots \text{ to } n \text{ factors.}$$

Theorem 2. Let $r_1, r_2, \cdots$ be the ranks of $(\alpha_\epsilon I + B)^n$ for $n = 1, 2, \cdots$; then the exponents of the elementary divisors of $\lambda I + B$ which are connected with the linear factor $\lambda - \alpha_\epsilon$ are uniquely determined by the following table:

$m - r_\kappa$	e_1	e_2	$\cdots$	e_π	$\cdots$	e_τ
$r_1 - r_\kappa$	$e_1 - 1$	$e_2 - 1$	$\cdots$	$e_\pi - 1$	$\cdots$	$e_\tau - 1$
$r_2 - r_\kappa$	$e_1 - 2$	$e_2 - 2$	$\cdots$	$e_\pi - 2$	$\cdots$	$e_\tau - 2$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

which is completely defined as follows: The order of $\lambda I + B$ is m and

I. There are κ rows where $r_1 > r_2 > r_3 > \cdots > r_{\kappa-1} > r_\kappa = r_{\kappa+1}$.

II. In any row the sum of the integers appearing is equal to the number at the left end of that row.

III. In the bottom row the integers appearing are all ones and are at the left $r_{\kappa-1} - r_\kappa$ in number. The spaces to the right are blank.

IV. In each successive row from the bottom upward each integer is one greater than that directly beneath it and a sufficient number of ones appear on the right to satisfy II.

Proof. By Theorem 1, the ranks $r_1, r_2, \cdots r_\kappa$ are the same for $(\alpha_\epsilon I + N)^n$. By the laws of matrix multiplication, it appears that $(\alpha_\epsilon I + N)^n$ is immediately obtained by writing $(\alpha_\epsilon I + N_{t_\epsilon})^n$ in place of $\alpha_\epsilon I + N_{t_\epsilon}$ in $\alpha_\epsilon I + N$ for $\epsilon = 1, 2, \cdots \eta$. In like manner $(\alpha_\epsilon I + N_{t_\epsilon})^n$ is obtained by writing $(\alpha_\epsilon I + E_{e_\pi})^n$ in place of $\alpha_\epsilon I + E_{e_\pi}$ in $\alpha_\epsilon I + N_{t_\epsilon}$

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* A complete and direct demonstration of this theorem is given by H. E. Hawkes in the American Journal of Mathematics, vol. XXXII, No. 2, pp. 101–109.

for $\pi = 1, 2, \cdots \tau$. Inspection of these results shows that the rank of $(\alpha_\epsilon I + N)^n$ is the sum of the ranks of its non-overlapping principal minors and that the same is true of $(\alpha_\epsilon I + N_{t_\epsilon})^n$. Further it is obvious that the sum of the ranks of all of the principal minors of $(\alpha_\epsilon I + N)^n$ excepting $(\alpha_\epsilon I + N_{t_\epsilon})^n$ is $m - t_\epsilon$; for their ranks do not change with n. Again if we compute $(\alpha_\epsilon I + E_{e_\pi})^n$, we see that, for $n = 1, 2, \cdots e_{\pi-1}, e_\pi$, the ranks are $e_\pi - 1, e_\pi - 2, \cdots 1, 0$. Since the ranks of these principal minors in $(\alpha_\epsilon I + N_{t_\epsilon})^n$ become zero as n increases through the succession $e_\tau \cdots e_\pi \cdots e_2, e_1$, we have $r_\kappa = m - t_\epsilon$ and $\kappa = e_1$. This establishes the theorem.

Illustration. Suppose we have a characteristic matrix of order 10 whose characteristic equation has $\lambda - \alpha$ as a linear factor, and that for this root α the ranks of $(\alpha I + B)^n$ are $r_1 = 6$, $r_2 = 4$, $r_3 = 3$, $r_4 = 3$; then $\kappa = 3$ and the table is

7	3	2	1	1
3	2	1		
1	1			

It thus appears that we need only to know the successive ranks $r_1 > r_2 > \cdots > r_{\kappa-1} > r_\kappa = r_{\kappa+1}$ of $(\alpha I + B)^n$ for $n = 1, 2, \cdots, \kappa - 1, \kappa, \kappa + 1$ in order to compute the exponents of the elementary divisors associated with the linear factor $\lambda - \alpha$. That this method is exceedingly simple in comparison with the ordinary method is apparent.

II. Matrices of the type $\lambda_1 A + \lambda_2 B$.

Case 1. If A is non-singular, we know that $\lambda_1 I + \lambda_2 A^{-1}B$ has the same elementary divisors as $\lambda_1 A + \lambda_2 B$.* Hence we determine the elementary divisors for the characteristic matrix $\lambda I + A^{-1}B$.

Case 2. If A and B arè both singular, but $|\lambda_1 A + \lambda_2 B| \neq 0$ identically in λ_1 and λ_2, the method is applicable. All that is necessary is to transform the parameters by a non-singular linear transformation in such a manner that the resulting matrix $\lambda_1' A' + \lambda_2' B'$ is brought under case 1. The exponents of the elementary divisors will be the same and the corresponding linear factors will be connected by the transformation of the parameters.†

Case 3. In case $|\lambda_1 A + \lambda_2 B| = 0$ identically in λ_1 and λ_2, the method can be applied provided the matrix can be reduced by elementary transformations to an equivalent matrix of lower order which falls under case 2.

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* Bôcher, Introduction to Higher Algebra, section94.

† Muth, Elementarteiler, p. 63.